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AN ANALYTIC SOLUTION OF A SYSTEM OF EQUATIONS FOR THE AXISYMMETRICAL FLOW OF AN INCOMPRESSIBLE FLUID

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G. I. NAZAROV and A. A. PUCHKOV
(Tomsk)
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In contrast to the solution of Garabedian [1] and to the solution constructed by the Bergman method [2], we derive an exact general solution for the pair of functions φ and ψ (φ is the velocity potential, ψ is the stream function) of a system of partial differential equations describing the axisymmetrical flow of an incompressible ideal fluid. Our solution depends on an arbitrary analytic function of a complex variable and is bounded on the axis of symmetry.

The solutions constructed in [1] and [2] increase without limit as the axis of symmetry is approached.

Three-dimensional steady-state axisymmetrical flows of an incompressible fluid are described by the system of Eqs. [3]

$$\frac{\partial\varphi}{\partial x} = -\frac{1}{y}\frac{\partial\psi}{\partial y}, \qquad \frac{\partial\varphi}{\partial y} = \frac{1}{y}\frac{\partial\psi}{\partial x} \tag{1}$$

Here the velocity potential φ and the stream function ψ depend on only the two variables x, y of the cylindrical coordinate system ($y > 0$ and x is parallel to the axis of symmetry).

The integrals of system (1) will be sought in series form

$$\varphi = \Omega(y) + \sum_{k=0}^{\infty} \alpha_k(y)\frac{\partial^k\Phi}{\partial y^k}, \qquad \psi = A + Bx + \sum_{k=0}^{\infty} \beta_k(y)\frac{\partial^k\Psi}{\partial y^k} \tag{2}$$

Here Φ, Ψ are arbitrary harmonic functions which satisfy the Cauchy-Riemann conditions

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y}, \qquad \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x} \tag{3}$$

where $\Omega, \alpha_k, \beta_k$ $(k = 0, 1, 2, \ldots)$ are the required functions of the single argument y.

Let us construct the corresponding derivatives of (2), substitute them into (1), and recall Eqs. (3) and relations of the form

$$\frac{\partial^{k+1}\Phi}{\partial x\partial y^k} = \frac{\partial^{k+1}\Psi}{\partial y^{k+1}}, \qquad \frac{\partial^{k+1}\Psi}{\partial x\,\partial y^k} = -\frac{\partial^{k+1}\Phi}{\partial y^{k+1}} \tag{4}$$

which follow from (3).

We then have

$$\sum_{k=0}^{\infty} \alpha_k \frac{\partial^{k+1} \Psi}{\partial y^{k+1}} = -\frac{1}{y} \sum_{k=0}^{\infty} \left(\beta_k' \frac{\partial^k \Psi}{\partial y^k} + \beta_k \frac{\partial^{k+1} \Psi}{\partial y^{k+1}} \right)$$

$$\Omega' + \sum_{k=0}^{\infty} \left(\alpha'_k \frac{\partial^k \Phi}{\partial y^k} + \alpha_k \frac{\partial^{k+1} \Phi}{\partial y^{k+1}} \right) = \frac{1}{y} \left(B - \sum_{k=0}^{\infty} \beta_k \frac{\partial^{k+1} \Phi}{\partial y^{k+1}} \right)$$

This system is satisfied if we impose conditions of the form

$$\alpha'_0 = 0, \quad \beta'_0 = 0, \quad \Omega' = \frac{B}{y}$$

$$\alpha_{k-1} = -\frac{1}{y} (\beta_k' + \beta_{k-1}), \quad \alpha_k' + \alpha_{k-1} = -\frac{\beta_{k-1}}{y} \; (k = 1, 2, \ldots) \tag{5}$$

From this we see that $\alpha_0 = \alpha$ and $\beta_0 = \beta$ can only be arbitrary constants, and that α_k and β_k ($k = 1, 2, \ldots$) can be expressed as a combination of the preceding functions α_{k-1}, β_{k-1}. Ultimately, all of these functions can be expressed in terms of just α, β and y and tabulated once and for all by means of Formulas

$$\alpha_k = -\int_{y_0}^{y} \left(\frac{\beta_{k-1}}{y} + \alpha_{k-1} \right) dy, \quad \beta_k = -\int_{y_0}^{y} (y\alpha_{k-1} + \beta_{k-1})\, dy \quad (k = 1, 2, \ldots) \tag{6}$$

Let us isolate the class of solutions, which in contrast to the classes of [1 and 2] does not become infinite on the axis of symmetry ($y = 0$). To this end we arbitrarily set $B = 0$, $\beta_0 = 0$ and $\beta = 0$. In this case we find from (5) that $\Omega = c = \text{const}$. Without limiting generality we can set $\alpha_0 = \alpha = 1$.

Taking the lower limit $y_0 = 0$ in (6), we arrive at the recurrent relations

$$\alpha_k = a_k y^k, \qquad \beta_k = b_k y^{k+1} \quad (k = 0, 1, 2, \ldots) \tag{7}$$

$$a_k = \frac{k+1}{k} b_k, \; b_k = -\frac{1}{k+1} (a_{k-1} + b_{k-1}) \quad (a_0 = 1, \; b_0 = 0) \tag{8}$$

From (2), (7) and (8) we obtain Formulas

$$\varphi = \sum_{k=0}^{\infty} a_k y^k \frac{\partial^k \Phi}{\partial y^k}, \quad \psi = \sum_{n=1}^{\infty} b_n y^{n+1} \frac{\partial^n \Psi}{\partial y^n} \tag{9}$$

$$a_k = (-1)^{k+1} \frac{(2k-1)!!}{k!k!}, \quad b_n = (-1)^n \frac{(2n-1)!!}{(n-1)!(n+1)!}$$

$$(k = 0, 1, 2, \ldots; \quad n = 1, 2, \ldots, \quad (-1)!! = -1) \tag{10}$$

Formulas (9) can be written in complex form as

$$\varphi = \operatorname{Re} \sum_{k=0}^{\infty} a_k (iy)^k \frac{d^k w}{dz^k}, \quad \psi = y \operatorname{Im} \sum_{n=1}^{\infty} b_n (iy)^n \frac{d^n w}{dz^n} \tag{11}$$

Here $w(z) = \Phi + i\Psi$ is an arbitrary analytic function of the complex variable $z = x + iy$. We have the familiar Cauchy formula

$$\frac{d^k w}{dz^k} = \frac{k!}{2\pi i} \int_{\gamma} \frac{w(\zeta)\, d\zeta}{(\zeta - z)^{k+1}} \tag{12}$$

(γ is an arbitrary closed contour within which the point z lies).

We substitute (12) into (11) and, assuming that the series converge uniformly, interchange the summation and integration symbols (the domain of convergence will be indicated below).

Taking account of (10) we obtain

$$\varphi = -\operatorname{Re}\frac{1}{2\pi i}\int_\gamma \frac{w(\zeta)}{\zeta - z}\sum_{k=0}^{\infty} C_k\left(\frac{2iy}{z-\zeta}\right)^k d\zeta, \quad \left(C_k = \frac{(2k-1)!!}{k!\,2^k}\right)$$

$$\psi = -y^2 \operatorname{Im}\frac{1}{\pi}\int_\gamma \frac{w(z)}{(\zeta - z)^2}\sum_{n=1}^{\infty} nD_n\left(\frac{2iy}{z-\zeta}\right)^{n-1} d\zeta, \quad \left(D_n = \frac{(2n-1)!!}{(n+1)!\,2^n}\right) \tag{13}$$

We note that the coefficients

$$C_{k+1} = \frac{(k+{}^1/_2)(k+1)}{(k+1)(k+1)}C_k, \qquad D_{n+1} = \frac{(n+{}^1/_2)(n+1)}{(n+1)(n+2)}D_n$$

vary as in the hypergeometric series. This enables us to write Formulas (13) as

$$\varphi = -\operatorname{Re}\frac{1}{2\pi i}\int_\gamma \frac{w(\zeta)}{\zeta - z}F\left({}^1/_2, 1, 1, \frac{2yi}{z-\zeta}\right)d\zeta$$

$$\psi = -y\operatorname{Im}\frac{1}{2\pi i}\int_\gamma w(\zeta)\frac{d}{d\zeta}F\left({}^1/_2, 1, 2, \frac{2yi}{z-\zeta}\right)d\zeta \tag{14}$$

From (14) we can readily see that the series appearing in (11) converge in the domain

$$\left|\frac{2yi}{z-\zeta}\right| \leqslant a < 1$$

Or, setting $\rho = |z - \zeta|$, we have $\rho > 2y$ $(y > 0)$, i.e.

$$(\xi - x)^2 + (\eta - y)^2 > 4y^2 \quad (z = x + iy,\ \zeta = \xi + i\eta) \tag{15}$$

Condition (15) is always satisfied on the line $y = 0$. From (14) we find that $\psi = 0$ for $y = 0$ (this is the zero streamline). Integrals (9) or (which is the same thing) (11) and (14) are bounded on the axis of symmetry. This makes them markedly different from the known solutions of Garabedian [1] and of [2].

We have therefore proved the following theorem: by way of linear operators (11) or (14) the complex potential $w(z)$ defines some axisymmetrical flow of an ideal incompressible fluid.

Specifying an arbitrary analytic function $w(z)$ in (11), we can obtain various particular solutions and thus assemble a collection of elementary axisymmetrical flows of an incompressible fluid.

By virtue of the linearity of system (1), any linear combination of particular solutions is also a solution. This approach is analogous to the well-known inverse method of the theory of plane incompressible fluid flow. On the other hand, in solving direct boundary value problems the function $w(z)$ must be found by some method. The latter problem is incomparably more difficult than the former.

Let us cite several examples based on the choice of the function $w(z)$.

1. Let us set $w(z) = Uz$ (where U is a real constant). We then find from (10), (11) that

$$\varphi = Ux, \quad \psi = -{}^1/_2 Uy^2$$

These are the familiar formulas characterizing a homogeneous axisymmetrical stream moving with the velocity U parallel to the axis of symmetry [3].

2. Let us set $w = Q/z$ (a plane dipole). Then

$$w^{(k)}(z) = Q(-1)^k \frac{k!}{z^{k+1}} \tag{16}$$

For function (16) Formulas (10) and (11) yield

$$\varphi = -Q \operatorname{Re} \frac{1}{z} \sum_{k=0}^{\infty} C_k \zeta^k, \quad \psi = -2y^2 \operatorname{Im} \frac{i}{z^2} \sum_{n=1}^{\infty} n D_n \zeta^{n-1} \tag{17}$$

Here

$$C_k = \frac{(2k-1)!!}{k!\,2^k}, \quad D_n = \frac{(2n-1)!!}{(n+1)!\,2^n}, \quad \zeta = \frac{2iy}{z} \tag{18}$$

Taking account of (18), we can rewrite Formulas (17) as

$$\varphi = -Q \operatorname{Re} \frac{1}{z} F(1/2, 1, 1, \zeta), \qquad \psi = -2y^2 \operatorname{Im} \frac{i}{z^2} \frac{d}{d\zeta} F\left(\frac{1}{2}, 1, 2, \zeta\right) \tag{19}$$

The following Eqs. are known [4]:

$$F\left(\frac{1}{2}, 1, 1, \sin^2 \xi\right) = \frac{1}{\cos \xi}, \quad F\left(\alpha - \frac{1}{2}, \alpha, 2\alpha, \zeta\right) = \left(\frac{2}{1 + \sqrt{1-\zeta}}\right)^{2\alpha - 1} \tag{20}$$

In this case $\zeta = \sin^2 \xi, \alpha = 1$. Then

$$\cos \xi = \sqrt{1-\zeta} = \sqrt{1 - \frac{2iy}{z}} = \frac{\sqrt{x^2+y^2}}{z}$$

$$\frac{dF}{d\zeta} = \frac{1}{\sqrt{1-\zeta}\,(1+\sqrt{1-\zeta})^2} = \frac{z^3}{\sqrt{x^2+y^2}\,(z + \sqrt{x^2+y^2})^2} \tag{21}$$

After simplification Formulas (19) to (21) become ($Q = q/4\pi$)

$$\varphi = -\frac{q}{4\pi} \frac{1}{\sqrt{x^2+y^2}}, \quad \psi = \frac{q}{4\pi}\left(\frac{x}{\sqrt{x^2+y^2}} - 1\right) \tag{22}$$

Formulas (22) characterize a three-dimensional source of strength q situated at the origin of the cylindrical coordinate system [3].

Linear operators (11), (14) with the complex potential $w = Uz + q/4\pi z$ describe flow past a Fuhrmann half-body (a semi-infinite solid of revolution) [3].

3. If $w = z^2$, then (10) and (11) readily yield Formulas

$$\varphi = x^2 - \frac{y^2}{2}, \qquad \psi = -xy^2 \tag{23}$$

In the plane case the function $w = z^2$ characterizes flow in the coordinate angles along equilateral hyperbolas with the velocity $v = 2\sqrt{x^2+y^2}$, at the point z ; as we see from (23), in the axisymmetrical case this function likewise yields flow past an angle in the meridional section x, y, although the streamlines in this case take the form of nonequilateral hyprebolas, and the velocity of the point z is $v = \sqrt{4x^2+y^2}$.

These examples show that there is no direct connection between axisymmetrical and plane flow of incompressible fluids: the same complex potential is generally associated with dissimilar flows.

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